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June 18^{th}, 2024

Lecture 16

§ 7.4 Integration of Rational functions by partial fractions.

Rational function : $f(x) = \dfrac{p(x)}{q(x)}$, $p(x), q(x)$: polynomials.

Long division

$\Rightarrow f(x) = s(x) + \dfrac{r(x)}{q(x)}$, $r(x)$: polynomial

$s(x)$: polynomial

such that $\deg r(x) < \deg q(x)$.

Example: $f(x) = \underbrace{2x+2}_{s(x)} + \dfrac{2x}{3x^2+5}$ $\leftarrow r(x)$, $\leftarrow q(x)$

We consider $f(x) = \dfrac{r(x)}{q(x)}$ with $\deg r(x) < \deg q(x)$

Evaluate $\displaystyle\int f(x)\,dx = \int \frac{r(x)}{q(x)}\,dx$.

Example: $\displaystyle\int \frac{2x}{(x-1)(2x-3)}\,dx$, $\displaystyle\int \frac{2x+5}{(x-3)^2(x+1)(2x-1)}\,dx$.

Method: Partial fractions

Example: $f(x) = \dfrac{2x}{(x-1)(2x-3)} = \dfrac{A}{x-1} + \dfrac{B}{2x-3}$

partial fraction

$\Rightarrow$ Goal: We want to write $f(x)$ into sum of partial fractions.

why?

write $f(x) = \dfrac{2x}{(x-1)(2x-3)} = \dfrac{A}{x-1} + \dfrac{B}{2x-3}$ $(*)$

then

$\displaystyle\int \frac{2x}{\ldots}\,dx = \int \Big(\ldots + \ldots \Big)\,dx$

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) into sum of ... partial fractions.

why?

Write. $f(x) = \frac{2x}{(x-1)(2x-3)} = \frac{A}{x-1} + \frac{B}{2x-3}$ (*)

then

$$\int \frac{2x}{(x-1)(2x-3)}\,dx \overset{(*)}{=} \int \left(\frac{A}{x-1} + \frac{B}{2x-3}\right) dx$$

Hard to compute

Easy to compute.

$$= A\int \frac{1}{x-1}\,dx + B\int \frac{1}{2x-3}\,dx$$

$$= A \ln|x-1| + \frac{B}{2} \ln|2x-3| + C$$

Goal: How to write $f(x)$ in terms of sum of partial fractions.

We have 4 cases.

Case I $q(x) = (a_1x+b_1)(a_2x+b_2)\dots(a_nx+b_n)$

$q(x)$ is the product of distinct linear factors.

Example: $q(x) = (x-1)(2x-3) \Rightarrow$ Case I.

$q(x) = (x-1)^2 = (x-1)(x-1)$ $\Rightarrow$ Not case I

$q(x) = (x^2+2)(x-1)(2x-3) \Rightarrow$ not case I.

(x^2+2) not a linear factor

In case I: we write.

$$\frac{r(x)}{q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \dots + \frac{A_n}{a_nx+b_n}$$

factors of $q(x)$

$A_1, A_2 \dots A_n$: constants.

Ex:

a) $\frac{r(x)}{} = \frac{2x}{}$

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$a_n x + b_n$

$q(x)$

$A_1, A_2 \ldots A_n$: constants.

Ex:

a) $\dfrac{r(x)}{q(x)} = \dfrac{2x}{(x-1)(2x-3)} = \dfrac{A}{(x-1)} + \dfrac{B}{2x-3}$

case 1

b) $\dfrac{r(x)}{q(x)} = \dfrac{2x^2+5}{(x-1)(x+2)(2x-5)(x+4)}$

case I

$$= \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{2x-5} + \frac{D}{x+4}$$

How to find A, B, C, D?

Method to find constants.

Multiply both sides with the least common denominators and set the numerators equal. $\Rightarrow$ solve for A, B, C....

Example: $\dfrac{r(x)}{q(x)} = \dfrac{2x}{(x-1)(2x-3)} = \dfrac{A}{x-1} + \dfrac{B}{2x-3}$

Multiply both side

$$\Rightarrow \quad 2x = A\,(2x-3) + B\,(x-1)$$

Expand $\Rightarrow \quad 2x = 2Ax - 3A + Bx - B$

$$\Rightarrow \quad 2x + 0 = x(2A+B) + (-3A-B)$$

Since RHS = LHS

$\Rightarrow$ coefficients must be equal.

$$\Rightarrow \begin{cases} 2 = 2A + B \\ \ldots \end{cases} \quad \Rightarrow \begin{cases} 2A + B = \\ \ldots \end{cases}$$

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$\Rightarrow \quad 2x + 0 = x(2A+B) + (-3A-B)$

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$\Rightarrow$ Coefficients must be equal.

$$\Rightarrow \begin{cases} 2 = 2A+B \\ 0 = -3A-B \end{cases} \Rightarrow \begin{cases} 2A+B = 2 & (1) \\ 3A+B = 0 & (2) \end{cases}$$

$*\Rightarrow$ Solve the system of equations to get A, B.

$$\Rightarrow \begin{cases} B = -3A \text{ (from the second line)} \\ 2A + (-3A) = 2 \end{cases}$$

$$\Rightarrow \begin{cases} B = -3A \\ -A = 2 \end{cases} \Rightarrow \begin{cases} B = -3A \\ A = -2 \end{cases} \Rightarrow \begin{cases} B = 6 \\ A = -2 \end{cases}$$

Conclusion.

$$f(x) = \frac{r(x)}{q(x)} = \frac{2x}{(x-1)(2x-3)} = \frac{\overset{A}{-2}}{x-1} + \frac{6}{2x-3}$$

Hence $\displaystyle\int f(x)\,dx = \int \frac{-2}{x-1}\,dx + \int \frac{6}{2x-3}\,dx$

$$= -2\ln|x-1| + \frac{6}{2}\ln|2x-3| + C.$$

Case II $\quad q(x) = (a_1x+b_1)^{r_1}(a_2x+b_2)^{r_2}\cdots(a_nx+b_n)^{r_n}$

$q(x)$ is the product of linear factors, some of them repeated.

Example: $q(x) = (x-1)^2(2x-3) \rightarrow$ case II

$= (x-1)(x-1)(2x-3)$

Write $f(x) = \dfrac{r(x)}{q(x)}$ in terms of sum of partial fractions.

$$f(x) = \frac{r(x)}{q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_{r_1}}{(a_1x+b_1)^{r_1}}$$

$$+ \frac{B_1}{(a_2x+b_2)} + \cdots + \frac{B_{r_2}}{(a_2x+b_2)^{r_2}}$$

.....

Example:

$f(x) = \dfrac{2x}{(x-1} \quad A \quad B \quad C$

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$(a_2 x + b_2)^{r_2}$

Example:

a) $f(x) = \dfrac{2x}{(x-1)^2(2x-3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-1)^2} + \dfrac{C}{2x-3}$

b) $f(x) = \dfrac{2x^2+5x+5}{(x-1)^3(2x-1)^2x^5}$

$= \dfrac{A}{(x-1)} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x-1)^3}$

$+ \dfrac{D}{(2x-1)} + \dfrac{E}{(2x-1)^2} +$

$+ \dfrac{F}{x} + \dfrac{G}{x^2} + \dfrac{H}{x^3} + \dfrac{K}{x^4} + \dfrac{L}{x^5}$

Solve a):

$f(x) = \dfrac{2x}{(x-1)^2(2x-3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-1)^2} + \dfrac{C}{2x-3}$

Multiply with least common denominator:

$2x = A(x-1)(2x-3) + B(2x-3) + C(x-1)^2$

$\Rightarrow 2x = A(2x^2-5x+3) + 2Bx - 3B + C(x^2-2x+1)$

$= 2Ax^2 - 5Ax + 3A + 2Bx - 3B + Cx^2 - 2Cx + C$

$\Rightarrow 0x^2 + 2x + 0.1 = x^2(2A+C) + x(-5A+2B-2C) + 1(3A-3B+C)$

LHS = RHS

$\Rightarrow$ coefficients are equal

$\Rightarrow \begin{cases} 2A + C = 0 & (1) \\ -5A+2B-2C = 2 & (2) \\ 3A-3B+C = 0 & (3) \end{cases}$

Solve the syst...

From (1) $\Rightarrow$

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$\Rightarrow$ coefficients are equal

$$\Rightarrow \begin{cases} 2A + C = 0 & (1) \\ -5A + 2B - 2C = 2 & (2) \\ 3A - 3B + C = 0 & (3) \end{cases}$$

+ Solve the system of equations.

From (1) $\Rightarrow$ $C = -2A$.

$$(2), (3) \Rightarrow \begin{cases} -5A + 2B - 2(-2A) = 2 \\ 3A - 3B + (-2A) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -A + 2B = 2 \\ A - 3B = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -3B + 2B = 2 \\ A = 3B \end{cases}$$

$$\Rightarrow \begin{cases} B = -2 \\ A = -6 \\ C = -2A = 12 \end{cases}$$

Conclusion:

$$f(x) = \frac{r(x)}{q(x)} = \frac{2x}{(x-1)^2(2x-3)} = \frac{-6}{x-1} + \frac{-2}{(x-1)^2} + \frac{12}{2x-3}.$$

Thus $$\int f(x)\,dx = \int \frac{-6}{x-1}\,dx + \int \frac{-2}{(x-1)^2}\,dx + \int \frac{12}{2x-3}\,dx$$

$$= -6\ln|x-1| + 2\frac{1}{x-1} + \frac{12}{2}\ln|2x-3| + C.$$

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Case III. $q(x)$ contains irreducible quadratic factors non of which is repeated.

or: $q(x)$ contains (ax^2+bx+c).

Example: $f(x) = \dfrac{r(x)}{q(x)} = \dfrac{(2x+5)(x-1)}{(x^2+1)(x^2+2)(x-2)}$

$$q(x) = \underbrace{(x^2+1)(x^2+2)}_{\text{quadratic factors}}(x-2) \Rightarrow \text{case III}$$

* In case III, we write $f(x)$ as sum of terms where the term has the form

$$\frac{Ax+B}{ax^2+bx+c}.$$

Example..

$$f(x) = \frac{(2x+5)(x-1)}{(x^2+1)(x^2+2)(x-2)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+2}$$

linear.

quadratic factors.

* Solve for constants A, B, C, ...

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Example

quadratic factor.

$$\int \frac{x^2+5x+1}{(x^2+1)(x-2)}\,dx \quad , \quad q(x) = (x^2+1)(x-2)$$

is in case III

Write $\dfrac{x^2+5x+1}{(x^2+1)(x-2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-2}$

Multiply with common denominator, we have.

$$x^2+5x+1 = (Ax+B)(x-2) + C(x^2+1)$$

$$= Ax^2 - 2Ax + Bx - 2B + Cx^2 + C$$

$$\Rightarrow 1x^2 + 5x + 1 = x^2(A+C) + x(-2A+B) + 1(-2B+C)$$

$$\Rightarrow \begin{cases} A + C = 1 \quad (1) \\ -2A + B = 5 \\ -2B + C = 1 \end{cases}$$

We can not write

$A+C = 1 = -2B+C$.

(wrong)!

$(1) \Rightarrow A = 1 - C.$

$$\Rightarrow \begin{cases} -2(1-C) + B = 5 \\ -2B + C = 1 \end{cases} \Rightarrow \begin{cases} -2B = 1 - C. \\ -2(1-C) + \frac{-1}{2}(1-C) = 5 \end{cases}$$

$$\Rightarrow \begin{cases} B = \frac{1-C}{-2} \\ -2 + 2C - \frac{1}{2} + \frac{1}{2}C = 5. \end{cases}$$

$$\Rightarrow \begin{cases} B = \frac{1-C}{-2} \\ \frac{5}{2}C = 5 + \frac{5}{2} \end{cases} \Rightarrow \begin{cases} B = \frac{1-C}{-2} \\ C = 3 \end{cases} \Rightarrow \begin{cases} B = 1 \\ C = 3 \end{cases}$$

$$\Rightarrow \begin{cases} A = 1 - C = -2. \\ B = 1 \\ C = 3 \end{cases}$$

Conclusion:

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Conclusion:

$$f(x) = \frac{x^2+5x+1}{(x^2+1)(x-2)} = \frac{-2x+1}{x^2+1} + \frac{3}{x-2}$$

Thus $\int f(x)\,dx = \int \frac{-2x+1}{x^2+1}\,dx + \underbrace{\int \frac{3}{x-2}\,dx}_{= 3\ln|x-2| + C.}$

Compute $\int \frac{-2x+1}{x^2+1}\,dx = \underbrace{-\int \frac{2x}{x^2+1}\,dx}_{1} + \underbrace{\int \frac{dx}{x^2+1}}_{2}$

Put $u = x^2+1 \Rightarrow du = 2x\,dx.$

Thus 1 $= -\int \frac{2x\,dx}{x^2+1} = -\int \frac{du}{u} = -\ln|u| + C = -\ln|x^2+1| + C.$

2 $= \int \frac{dx}{x^2+1} = \tan^{-1} x + C.$

Hence $\int f(x)\,dx = \int \frac{x^2+5x+1}{(x^2+1)(x-2)}\,dx = 3\ln|x-2| + \tan^{-1}(x) - \ln|x^2+1| + C$

* Rationalizing substitutions.

In some case, you can use substitution
$\Rightarrow$ rational function

Example: Evaluate $\int \frac{\sqrt{x+4}}{x}\,dx.$

Use substitution:
$u = \sqrt{x+4} \Rightarrow du = \frac{1}{2}(x+4)^{-1/2}\,dx.$
$\Rightarrow du = \frac{1}{2u}\,dx$
$\Rightarrow dx = 2u\,du$

$u = \sqrt{x+4} \Rightarrow u^2 - 4 = x.$

Then
$I = \int \frac{\sqrt{x+4}}{x}\,dx = 2\int \frac{u^2}{u^2-4}\,du$

we need deg $r(x) <$ deg $q(x)$
long division: $\frac{u^2}{u^2-4} = 1 + \frac{4}{u^2-4} = \frac{u^2-4}{u^2-4} + \frac{4}{u^2-4} = \frac{u^2}{u^2-4}$

$\Rightarrow I = 2\int \left(1 + \frac{4}{u^2-4}\right) du = 2\int du + 8\int \frac{1}{u^2-4}\,du$
$= 2u + 8\int \frac{1}{u^2-4}\,du$

Compute $\int \frac{1}{u^2-4}\,du = \int \frac{1}{(u-2)(u+2)}\,du$

distinct lin...

write $\frac{1}{(u-2)(u+2)} = \frac{}{u-2} \quad \frac{}{u+2}$

we need deg $r(x) <$ deg $q(x)$

$$\ldots = \frac{u^2-4}{u^2-4} + \frac{4}{u^2-4} = \frac{u^2}{u^2-4}$$

$$\Rightarrow I = 2\int\left(1+\frac{4}{u^2-4}\right)du = 2\int du + 8\int\frac{1}{u^2-4}du$$
$$= 2u + 8\int\frac{1}{u^2-4}du$$

Compute $\displaystyle\int\frac{1}{u^2-4}du = \int\frac{1}{(u-2)(u+2)}du$

distinct linear factors $\Rightarrow$ case 1.

Write $\displaystyle\frac{1}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2}$

Multiply with common denominator, we have

$$1 = \frac{A}{u-2}(u-2)(u+2) + \frac{B}{u+2}(u-2)(u+2)$$
$$\Rightarrow 1 = A(u+2) + B(u-2)$$
$$\Rightarrow 1 = Au + 2A + Bu - 2B$$
$$\Rightarrow 0.u + 1 = u(A+B) + 2A - 2B.$$
$$\Rightarrow \begin{cases} A+B = 0 \\ 2A-2B = 1 \end{cases} \Rightarrow \begin{cases} A = -B \\ 2(-B) - 2B = 1 \end{cases}$$
$$\Rightarrow \begin{cases} A = -B \\ -4B = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{4} \\ B = -\frac{1}{4} \end{cases}$$

$$\int\frac{1}{(u-2)(u+2)}du = \frac{1}{4}\int\frac{1}{u-2}du - \frac{1}{4}\int\frac{1}{u+2}du.$$
$$= \frac{1}{4}\ln|u-2| - \frac{1}{4}\ln|u+2| + C.$$

Conclusion

$$\int\frac{\sqrt{x+4}}{x}dx = 2u + \frac{8}{4}\ln|u-2| - \frac{8}{4}\ln|u+2| + C$$
$$= 2u + 2\ln|u-2| - 2\ln|u+2| + C$$
$$\underset{u=\sqrt{x+4}}{=} 2\sqrt{x+4} + 2\ln|\sqrt{x+4}-2| - 2\ln|\sqrt{x+4}+2| + C$$